ADDENDUM

"Random Excitation of Cylindrical Shells

Due to Jet Noise"

July, 1962

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Addendum to
National Aeronautics and Space Administration
Goddard Space Flight Center
Contract No. NAS 5-1515
Technical Report No. 1
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By

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ACKNOWLEDGEMENTS

This work was done under the sponsorship of Goddard Space Flight Center, NASA. The technical supervisor of the project is Mr. E. J. Kirchman. Many helpful suggestions were offered by Mr. Kirchman and Dr. E. Klein during discussions concerning the project. The author would also like to acknowledge the assistance of Mr. W. Bangs of Goddard Space Flight Center.

The data for the Saturn Tank was relayed to the author by Mr. G. A. Wilhold from Marshall Space Flight Center.

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LIST OF SYMBOLS

u_{mn} longitudinal displacement in mn^{th} mode

v_{mn} tangential displacement in mn^{th} mode

w_{mn} radial displacement in mn^{th} mode

A_{mn} longitudinal displacement amplitude

B_{mn} tangential displacement amplitude

C_{mn} radial displacement amplitude

m number of axial half waves in vibration pattern

n number of circumferential waves in vibration pattern

ω frequency of vibration

t time

x longitudinal coordinate

ϕ peripheral coordinate

a radius of the cylinder

E modulus of elasticity of the shell material

ν Poisson's ratio for the shell material

h thickness of the shell wall

l length of the shell between supports

H_u, H_v, H_w displacement transfer functions for longitudinal, tangential and radial displacement respectively

$$\lambda = \frac{m\pi a}{l}$$

$$q_1 = \frac{pa(1-\nu^2)}{Eh}$$

$$q_2 = \frac{1}{2}\frac{pa(1-\nu^2)}{Eh}$$

p internal pressure

θ_{mn} resistive impedance of fluid external to the shell

χ_{mn} reactive impedance of fluid external to the shell

γ_{mn} reactive impedance of internal fluid

$\alpha = h/a$

$\Omega = \omega a / C_p$; $C_p = \sqrt{\frac{E}{\rho_t(1-\nu^2)}}$

ρ_o mass density of outside medium

ρ_i mass density of internal fluid

δ damping parameter

C_i sound velocity of fluid inside shell

C_o sound velocity in medium surrounding shell

mass density of shell material

$C_r = \sqrt{\frac{E}{2\rho_t(1+\nu)}}$

$k\frac{\partial u}{\partial t}, k\frac{\partial v}{\partial t}, k\frac{\partial w}{\partial t}$ structural damping forces per unit area in the directions respectively

r radial direction

F_x, F_ϕ, F_r applied pressures in the x, ϕ, r directions respectively

p_{f_i} internal fluid pressure

p_{f_o} external fluid pressure

u, v, w total displacements in the x, ϕ, r directions respectively

I. Introduction

At certain stages of the flight a missile structure is exposed to intense jet noise. The thin walled tanks of a missile are pressurized close to the yield point of the material, thus the additional stresses induced by the jet noise could easily produce failure of the tanks. Since the static design of the tanks is so marginal, it is necessary to know how much additional dynamic stress is induced by way of the jet noise. The payloads are connected through attachments to the missile casing. Thus violent vibrations of the shell can result in violent motions of the payload unless enough is known about the vibration environment so that proper mounting of payloads can be achieved.

A limited amount of test data exists on response of the missile skin due to jet noise, but for the most part only theoretical predictions based on simplified theories[1,2] are readily available. This theoretical study has been divided into two main phases. The first phase which is being presented here consists of the computation of the missile shell response due to random loading. The second phase will be devoted to computation of the payload response using the shell motions as the input accelerations to the payload.

In predicting the response of shell structures one must be careful to consider all the important modes. In thin cylindrical shells the complication of the modal pattern is by no means a criterion for predicting the predominant modes as will be seen later in the report. For example, it is possible in thin cylindrical shells that for certain low frequencies the mode corresponding to twenty full waves around the periphery may be more important than the one corresponding to five. Each shell must be calculated separately and no general rule can be stated regarding the predominance of certain modes.

A frequency spectrum must first be computed to obtain the most important modes at given frequencies. The response functions can then be computed and the random response built up from these. Hand computations can be carried only so far with these complicated systems. It

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1. A. Powell, "On the Response of Structures to Random Pressures and to Jet Noise in Particular," Random Vibration, edited by S. Crandall, John Wiley and Sons, Inc., 1958, p. 187.

2. I. Dyer, "Estimation of Sound-Induced Missile Vibrations," Ibid, p. 231.

is neither efficient nor accurate to indulge in attempts to derive general formulas for the response when computer programs will determine response functions in fractions of a second. The computer programs must be developed which can be readily used by those involved in the dynamics of missiles. It is the purpose of this report to outline the theory for obtaining the frequency spectrum and response functions and then indicate how the programs can be used by those who are interested. The theory will then be worked out for part of the Saturn LOX tank and the results will be compared with tests run at the Marshall Space Flight Center.

II. Basic Equations

A. Response functions

For isotropic unstiffened pressurized shells containing fluid the equations of motion can be written[2a]

$$
\begin{aligned}
&a^2\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial \varphi^2}+\nu a\frac{\partial w}{\partial x}+\frac{1+\nu}{2}a\frac{\partial^2 v}{\partial x\partial\varphi}+\frac{h^2}{12a^2}\left[\frac{1-\nu}{2}\frac{\partial^2 u}{\partial\varphi^2}-a^3\frac{\partial^3 w}{\partial x^3}+\frac{1-\nu}{2}a\frac{\partial^3 w}{\partial x\partial\varphi^2}\right]\\
&\quad-\frac{\rho_t a^2}{E}(1-\nu^2)\frac{\partial^2 u}{\partial t^2}-\frac{a^2(1-\nu^2)}{Eh}k\frac{\partial u}{\partial t}=\frac{pa(1-\nu^2)}{Eh}\left(a\frac{\partial w}{\partial x}-\frac{\partial^2 u}{\partial\varphi^2}\right)-\frac{F_x a^2(1-\nu^2)}{Eh}\\
&\frac{1+\nu}{2}a\frac{\partial^2 u}{\partial x\partial\varphi}+\frac{\partial^2 v}{\partial\varphi^2}+\frac{1-\nu}{2}a^2\frac{\partial^2 v}{\partial x^2}+\frac{\partial w}{\partial\varphi}+\frac{h^2}{12a^2}\left[\frac{3(1-\nu)}{2}a^2\frac{\partial^2 v}{\partial x^2}-\frac{3-\nu}{2}a^2\frac{\partial^3 w}{\partial x^2\partial\varphi}\right]\\
&\quad-\frac{\rho_t a^2}{E}(1-\nu^2)\frac{\partial^2 v}{\partial t^2}-\frac{a^2(1-\nu^2)}{Eh}k\frac{\partial v}{\partial t}=-\frac{1}{2}\frac{pa(1-\nu^2)}{Eh}a^2\frac{\partial^2 v}{\partial x^2}-\frac{F_\varphi a^2(1-\nu^2)}{Eh} \qquad [1]\\
&\nu a\frac{\partial u}{\partial x}+\frac{\partial v}{\partial\varphi}+w+\frac{h^2}{12a^2}\left[\frac{1-\nu}{2}a\frac{\partial^3 u}{\partial x\partial\varphi^2}-a^3\frac{\partial^3 u}{\partial x^3}-\frac{3-\nu}{2}a^2\frac{\partial^3 v}{\partial x^2\partial\varphi}+a^4\frac{\partial^4 w}{\partial x^4}+2a^2\frac{\partial^4 w}{\partial x^2\partial\varphi^2}\right.\\
&\quad\left.+\frac{\partial^4 w}{\partial\varphi^4}+2\frac{\partial^2 w}{\partial\varphi^2}+w\right]+\frac{\rho_t a^2}{E}(1-\nu^2)\frac{\partial^2 w}{\partial t^2}+\frac{a^2(1-\nu^2)}{Eh}k\frac{\partial w}{\partial t}=\\
&\quad\frac{F_r a^2(1-\nu^2)}{Eh}+\frac{a^2(1-\nu^2)}{Eh}\left[p_{f_i}-p_{f_o}\right]+\frac{pa(1-\nu^2)}{Eh}\left[\frac{\partial^2 w}{\partial\varphi^2}+a\frac{\partial u}{\partial x}+w\right]+\frac{pa(1-\nu^2)}{2Eh}a^2\frac{\partial^2 w}{\partial x^2}
\end{aligned}
$$

2a. W. Flugge, "Statik und Dynamik der Schalen," Springer-Verlag, 1954, p. 191 and 229.

If the shell is assumed to have freely supported ends then the modal functions can be discribed by the set of displacements

$$
\begin{aligned}
u_{mn} &= A_{mn} \cos\frac{m\pi x}{\ell} \cos n\phi \, e^{i\omega t} \\
v_{mn} &= B_{mn} \sin\frac{m\pi x}{\ell} \sin n\phi \, e^{i\omega t} \\
w_{mn} &= C_{mn} \sin\frac{m\pi x}{\ell} \cos n\phi \, e^{i\omega t}
\end{aligned}
\qquad [2]
$$

where the nomenclature is given in the list of symbols.

These modal functions are very realistic for thin shells which are stiffened by rings. These functions describe the displacements between the rings assuming the rings to be rigid supports for the very thin shell. If the internal fluid pressure can be represented by the pressure that would be induced if the shell were extended indefinitely in both directions then the equations for the response functions derived in a previous reference[3] can be applied directly.

Consider first the stationary response to a simple harmonic function applied normal to the shell (in the same fashion as Crandall and Yildiz[4] for beams)

$$f(x, \phi, t) = e^{i\omega t} \cos n\phi \sin\frac{m\pi x}{\ell} \qquad [3]$$

The response will be the solution to the following set of simultaneous algebraic equations

$$
\begin{aligned}
A_{mn}[a_{11}+ib_{11}] + B_{mn}[a_{12}] + C_{mn}[a_{13}] &= 0 \\
A_{mn}[a_{21}] + B_{mn}[a_{22}+ib_{22}] + C_{mn}[a_{23}] &= 0 \\
A_{mn}[a_{31}] + B_{mn}[a_{32}] + C_{mn}[a_{33}+ib_{33}] &= \frac{a^2(1-\nu^2)}{Eh}
\end{aligned}
\qquad [4]
$$

where

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3. J. E. Greenspon, "Vibrations of Thick and Thin Cylindrical Shells Surrounded by Water," J G Engineering Research Associates, Baltimore, Maryland, Contract No. Nonr - 2733(00), Tech. Rep. No. 4, Sept. 1960 (Sponsored by Office of Naval Research)

4. S. H. Crandall and A. Yildiz, "Random Vibration of Beams," ASME Paper No. 61-WA-149.

$$a_{11} = -\lambda^2 - \frac{1-\nu}{2}n^2 - \frac{\alpha^2}{12}\frac{1-\nu}{2}n^2 + \Omega^2 - q_1 n^2$$

$$a_{12} = \frac{1+\nu}{2}\lambda n$$

$$a_{13} = \nu\lambda + \frac{\alpha^2}{12}\left(\lambda^3 - \frac{1-\nu}{2}n^2\lambda\right) - q_1\lambda$$

$$a_{21} = a_{12}$$

$$a_{22} = -n^2 - \frac{1-\nu}{2}\lambda^2 - \frac{\alpha^2}{12}\frac{3}{2}(1-\nu)\lambda^2 - q_2\lambda^2 + \Omega^2$$

$$a_{23} = -n - \frac{\alpha^2}{12}\frac{3-\nu}{2}n\lambda^2$$

$$a_{31} = a_{13}$$

$$a_{32} = a_{23}$$

$$a_{33} = -1 - \frac{\alpha^2}{12}(\lambda^4 + 2\lambda^2 n^2 + n^4 - 2n^2 + 1) + q_1(1-n^2) - q_2\lambda^2 + \Omega^2 + \chi' + \gamma'$$

$$b_{11} = -\Omega S$$

$$b_{22} = b_{11}$$

$$b_{33} = -\theta' - \Omega S$$

where $\lambda = \frac{m\pi a}{l}$, $\alpha = \frac{h}{a}$, $\Omega = \frac{\omega a}{c_p}$, $c_p = \sqrt{\frac{E}{\rho_t(1-\nu^2)}}$

$S = \frac{2\bar{n}a}{c_p} = \frac{K_a}{\rho h}\frac{1}{c_p}$, $\frac{\bar{n}}{\omega_{res}} = \frac{c'}{c'_c}$ (ratio of damping to critical damping)

$q_1 = \frac{pa(1-\nu^2)}{Eh}$, $q_2 = \frac{1}{2}q_1$, $\chi' = -\Omega\frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\chi_{mn}$

$\theta' = \Omega\frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\theta_{mn}$, $\gamma' = \Omega^2\frac{\rho_i}{\rho_t}\frac{a}{h}\gamma_{mn}$

For explanations of the impedances associated with the internal and external fluid ($\chi_{mn}, \vartheta_{mn}, \gamma_{mn}$) the reader is referred to a previous reference.[3]

The stationary response can be written

$$
\begin{aligned}
u(x,\varphi,t) &= H_u(m,n,\omega)\, e^{i\omega t} \cos\frac{m\pi x}{\ell} \cos n\varphi \\
v(x,\varphi,t) &= H_v(m,n,\omega)\, e^{i\omega t} \sin\frac{m\pi x}{\ell} \sin n\varphi \\
w(x,\varphi,t) &= H_w(m,n,\omega)\, e^{i\omega t} \sin\frac{m\pi x}{\ell} \cos n\varphi \\
H_u = A_{mn}\ , \quad & H_v = B_{mn}\ , \quad H_w = C_{mn}
\end{aligned}
\tag{5}
$$

The H's are known as the displacement transfer functions for lateral loading on the shell. These functions can be used for response due to lateral random loading. Similar response functions can be written for longitudinal and tangential loading.

The lateral loading on the shell can be expanded into the Fourier series

$$
f(x,\varphi,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} f_{mn}(t) \sin\frac{m\pi x}{\ell} \cos n\varphi
$$

where $$f_{mn}(t) = \frac{2}{\pi \ell}\int_0^{2\pi}\int_0^{\ell} f(x,\varphi,t) \sin\frac{m\pi x}{\ell} \cos n\varphi \, dx\, d\varphi \tag{6}$$

B. Frequencies

The frequency spectrum is obtained from the frequency equation

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} - \chi' - \gamma'
\end{vmatrix} = 0 \tag{7}
$$

The values of ω which satisfy the above determinant for a given $\lambda, n, h/a, \nu, q_1, q_2$ are the frequencies, ω_{res} of the shell for the given mode shape described by λ, n. The determinant will have three roots for each mode shape; usually for practical applications the lowest root will be the only one of significance.

Arnold and Warburton[5] have derived an analogous frequency equation using several simplifying assumptions. In essence they use the same displacement functions but employ Timoshenko's expressions for the strains[6] instead of Flugge's. It has been shown that the Arnold-

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5. R. N. Arnold and G. B. Warburton, Proc. Roy. Soc. A, Vol. 197, 1949, p. 238.

6. S. Timoshenko, "The Theory of Plates and Shells," McGraw Hill Book Co., Inc., 1940, p. 439.

Warburton theory is quite accurate even for thicker shells.[7] Their frequency equation is (unpressurized shell)

$$\Omega^3 - K_2\Omega^2 + K_1\Omega - K_0 = 0$$

where

$$K_0 = \tfrac{1}{2}(1-\nu)^2(1+\nu)\lambda^4 + \tfrac{1}{2}(1-\nu)\frac{h^2}{12a^2}\Big[(\lambda^2+n^2)^4 - 2(4-\nu^2)\lambda^4 n^2 - 8\lambda^2 n^4 - 2n^6 + 4(1-\nu^2)\lambda^4 + 4\lambda^2 n^2 + n^4\Big] \qquad [8]$$

$$K_1 = \tfrac{1}{2}(1-\nu)(\lambda^2+n^2)^2 + \tfrac{1}{2}(3-\nu-2\nu^2)\lambda^2 + \tfrac{1}{2}(1-\nu)n^2 + \frac{h^2}{12a^2}\Big[\tfrac{1}{2}(3-\nu)(\lambda^2+n^2)^3 + 2(1-\nu)\lambda^4 - (2-\nu^2)\lambda^2 n^2 - \tfrac{1}{2}(3+\nu)n^4 + 2(1-\nu)\lambda^2 + n^2\Big]$$

$$K_2 = 1 + \tfrac{1}{2}(3-\nu)(\lambda^2+n^2) + \frac{h^2}{12a^2}\Big[(\lambda^2+n^2)^2 + 2(1-\nu)\lambda^2 + n^2\Big]$$

Effect of Internal Pressure

There are a number of papers which discuss the effects of internal and external pressure on the frequencies of vibration of thin shells. Among the more extensive studies is that of Fung, Sechler, and Kaplan[8] which also contains a brief summary of some of the other important papers on pressurized shells. It can be shown however, that if one examines the secular determinant for free vibrations of freely supported pressurized shells the effect of internal or external pressure can immediately be written down without any further simplifications of the theory. To demonstrate this, the displacement expressions for freely supported ends are substituted into the Flugge shell equations;[9]

[7] J. E. Greenspon, J. Acoust. Soc. Am., 32, 571-578 (1960).

[8] Y. C. Fung, E. E. Sechler, A. Kaplan, J. Aero. Sci., 24, 650-660 (1960).

[9] W. Flugge, "Stresses in Shells," Springer-Verlag, 1960, p. 423.

there results

$$A_{mn}\left[\lambda^2+\frac{1-\nu}{2}n^2\left(1+\frac{h^2}{12a^2}\right)-q_1 n^2-q_2\lambda^2\right]+B_{mn}\left[-\frac{1+\nu}{2}\lambda n\right]$$
$$+C_{mn}\left[-\nu\lambda-\frac{h^2}{12a^2}\left(\lambda^3-\frac{1-\nu}{2}\lambda n^2\right)-q_1\lambda\right]=0$$

$$A_{mn}\left[-\frac{1+\nu}{2}\lambda n\right]+B_{mn}\left[n^2+\frac{1-\nu}{2}\lambda^2\left(1+3\frac{h^2}{12a^2}\right)-q_1 n^2-q_2\lambda^2\right]$$
$$+C_{mn}\left[n+\frac{3-\nu}{2}\frac{h^2}{12a^2}\lambda^2 n-q_1 n\right]=0 \qquad [9]$$

$$A_{mn}\left[-\nu\lambda-\frac{h^2}{12a^2}\left(\lambda^3-\frac{1-\nu}{2}\lambda n^2\right)-q_1\lambda\right]+B_{mn}\left[n+\frac{3-\nu}{2}\frac{h^2}{12a^2}\lambda^2 n-q_1 n\right]$$
$$+C_{mn}\left[1+\frac{h^2}{12a^2}\left(\lambda^4+2\lambda^2n^2+n^4-2n^2+1\right)-q_1 n^2-q_2\lambda^2\right]=0$$

In almost all practical cases $q_1, q_2 << \nu < 1$. Therefore all terms containing q_1 and q_2 can be neglected except those in the C bracket of the third equation above. However if we only neglect the q_1, q_2 terms in the off diagonal coefficients, the diagonal coefficients can be written

$$a_{11}-\bar{\Omega}^2 \quad , \quad a_{22}-\bar{\Omega}^2 \quad , \quad a_{33}-\bar{\Omega}^2 \qquad [10]$$

where

$$\bar{\Omega}^2=\Omega^2-q_1 n^2-q_2\lambda^2 \qquad [11]$$

Thus the natural frequency parameter of the pressurized shell is

$$\Omega=\bar{\Omega}\sqrt{1+\frac{q_1 n^2+q_2\lambda^2}{\bar{\Omega}^2}} \qquad [12]$$

where $\bar{\Omega}$ is merely the eigenvalue of the unpressurized shell. This result is quite general and will hold for shells of appreciable thickness since the Flugge and Arnold-Warburton equations do hold for such shells.[7] Equation [12] can also be shown to hold in the case of some anisotropic shells.

The results of equation [12] are in variance with those of Baron-Bleich[10] for n=1; however their results approach the values predicted by this theory for larger n. According to [12] an internal pressure will always give an increase in natural frequency and an external pressure will always result in a decrease no matter what mode is considered. Furthermore, the effects of internal pressure will only be felt for the n=0 mode if λ is fairly large. The formula of Fung et al[8] also predicts the exact pressure effect as [12]. This result is however more general than the frequency equation offered by these investigators.

C. Random loading

By a straight forward extension of the Crandall-Yildiz analysis for beams[4] the lateral displacement of the shell can be written

$$w(x,\phi,t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sin\frac{m\pi x}{l}\cos n\phi\int_{-\infty}^{+\infty}f_{mn}(\theta)h_w(m,n,t-\theta)\,d\theta \qquad [13]$$

where h_w is the lateral deflection due to a unit impulse at t=0.

If the load is assumed to have a known spatial distribution $g(x,\phi)$ and a random distribution in time $f(t)$, then the displacement at point x_1, ϕ_1 can be written

$$w(x_1,\phi_1,t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sin\frac{m\pi x_1}{l}\cos n\phi_1\int_{-\infty}^{+\infty}f(\theta)h_w(m,n,t-\theta)\,d\theta \left[\frac{2}{\pi l}\int_0^l\int_0^{2\pi}g(x,\phi)\sin\frac{m\pi x}{l}\cos n\phi\,dx\,d\phi\right] \qquad [14]$$

Now let

$$A_{mn}=\sin\frac{m\pi x_1}{l}\cos n\phi_1\,\frac{2}{\pi l}\int_0^l\int_0^{2\pi}g(x,\phi)\sin\frac{m\pi x}{l}\cos n\phi\,dx\,d\phi \qquad [15]$$

Then the deflection at x_1, ϕ_1 as a function of time will be

$$w(t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}A_{mn}\int_{-\infty}^{+\infty}f(\theta)h_w(m,n,t-\theta)\,d\theta \qquad [16]$$

From this point on all the well known theorems[11] for random loading of single degree of freedom systems can be applied to each term of the above series. In particular the power spectral density of the response S_w can be written

$$S_w(\omega)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2S_p(\omega)\left|H_w(m,n,\omega)\right|^2 \qquad [17]$$

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10. H. H. Bleich and M. L. Baron, J. Appl. Mech, 21, 167-177, 1954.

11. S. Crandall, "Statistical Properties of Response to Random Vibration," Random Vibration, John Wiley and Sons, Inc., 1958, p. 77.

where $S_p(\omega)$ is the power spectrum of the load.

The power spectra of the velocity and acceleration follow directly by simply multiplying by ω^2 and ω^4 respectively. The power spectra of the longitudinal and tangential response as well as the stresses is obtained by using the appropriate H .

D. Electronic computer programs

An electronic computer program has been developed for calculating the transfer functions of u, v, w and the longitudinal and tangential stresses at the surface of the cylinder. A computer program has also been setup for the natural frequencies of unpressurized shells by use of the Arnold-Warburton equations. The pressurized shell frequencies are obtained by formula [12].

III. Illustrative example - Saturn Lox Tank

A. General description

The Marshall Space Flight Center has reported results of static tests on the Saturn missile.[12] The test tank is a ring stiffened structure, however the rings are assumed stiff enough so that they act essentially as supports for the skin between them. The test tank is one of a series of peripheral tanks in the Saturn vehicle and therefore the entire tank is not exposed to the jet noise. The acoustically shaded portion of the tank is as shown in Figure 1. It will be assumed that the loading is random in time but constant in space for the exposed portion of the shell and zero for the shaded portion. The sound spectrum was extrapolated from measurements by Farrow et al.[12] This load spectrum at the test section is as shown in Figure 2.

B. Natural frequencies of the test shell

The frequencies of the unpressurized and pressurized test section were computed from the Arnold-Warburton frequency equation[5] and formula [12] given in this report. These frequency spectra are shown in Figures 3 and 4. One should note the great effect of pressure for large values of n. It is apparent from the frequency curves that the important modes over a given frequency band cannot arbitrarily be selected without careful study of the spectrum. Modes with large values of n may be very important at low frequencies. Pressure has such a large effect that modes which might

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12. J. H. Farrow, R. E. Jewell, and G. A. Wilhold, "Structural Response to the Noise Input of the Saturn Engines," Symposium on Structural Dynamics of High Speed Flight, April 24-26, 1961, p. 710.

be insignificant at a given frequency in an unpressurized shell may be predominant at that frequency in the actual pressurized tank.

C. Comparison between theoretical and experimental response

The acceleration pickup was located 11 inches from the lower ring as shown in Figure 1. The response functions were computed at this point. Assuming $\phi = \frac{\pi}{2}$ (see Fig. 1), the value of A_{mn} is

$$A_{mn} = \frac{8}{mn\pi^2} \sin \frac{m\pi x_1}{l}$$

(The pickup is located at $\phi = 0$)

Since the pressure was assumed constant over the length of the shell and constant over the range $-\phi \rightarrow +\phi$ $(\phi = \pi/2)$, only odd m and odd n modes are excited. No axially symmetric motions (n=0) will be excited by this load distribution.

The response was computed for 300, 400, 500, 600, 700, 800 cps. The modes considered at each frequency are given below in Table 1.

Table 1

Predominant Modes

Frequency	Modes Used in Calculations m	n
300 cps	1	1,3,5,7
400	1	1,3,5,7,9,11
500	1	1,3,13
600	1	1,3,5,15
700	3	9,11,13
800	1	1
	3	9,11,13

The following input parameters were used in the calculations:

$$\alpha = .0029,\ \nu = .3,\ \delta = .0063,\ g_1 = .0014,\ g_2 = .0007,$$
$$C_r/C_i = 10,\ C_r/C_o = 10,\ C_o/C_p = .062,\ \rho_o/\rho_t = .0005,\ \rho_i/\rho_t = .0005$$

The damping constant, $\delta = .0063$ was obtained from the experimental results of Fung and Sechler[8] on aluminum shells.

The experimental and theoretical acceleration spectra for a bandwidth of 10 cps are shown in Figure 5. The experimental results show a minor peak at about 150 cps whereas the theory predicts the lowest frequency at about 340 cps (see Fig. 4). This latter frequency corresponds to the first major peak shown in the experimental response curve. The low peak at 150 cps is probably a mode of the entire stiffened tank.

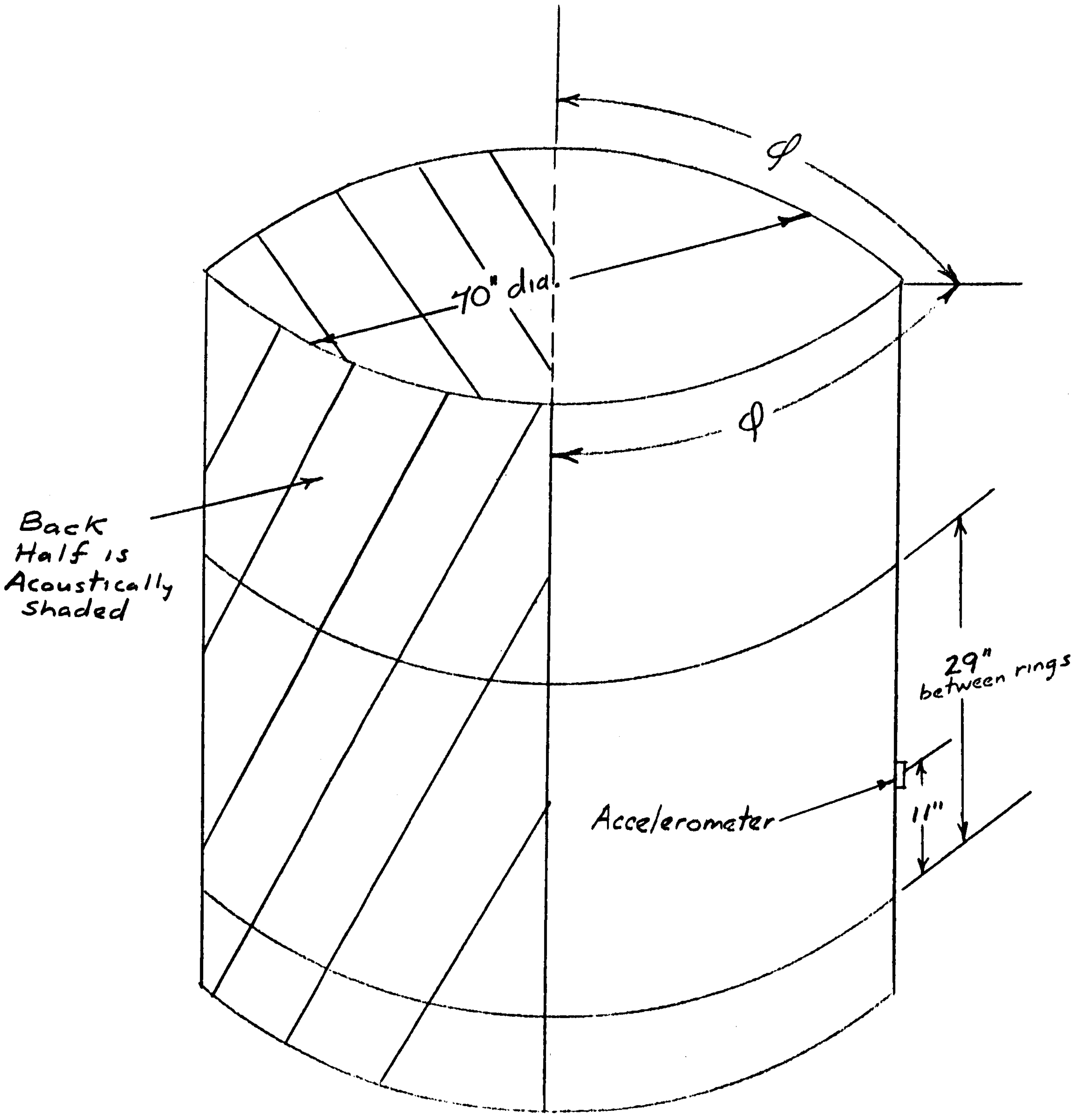

Fig 1. Test Shell

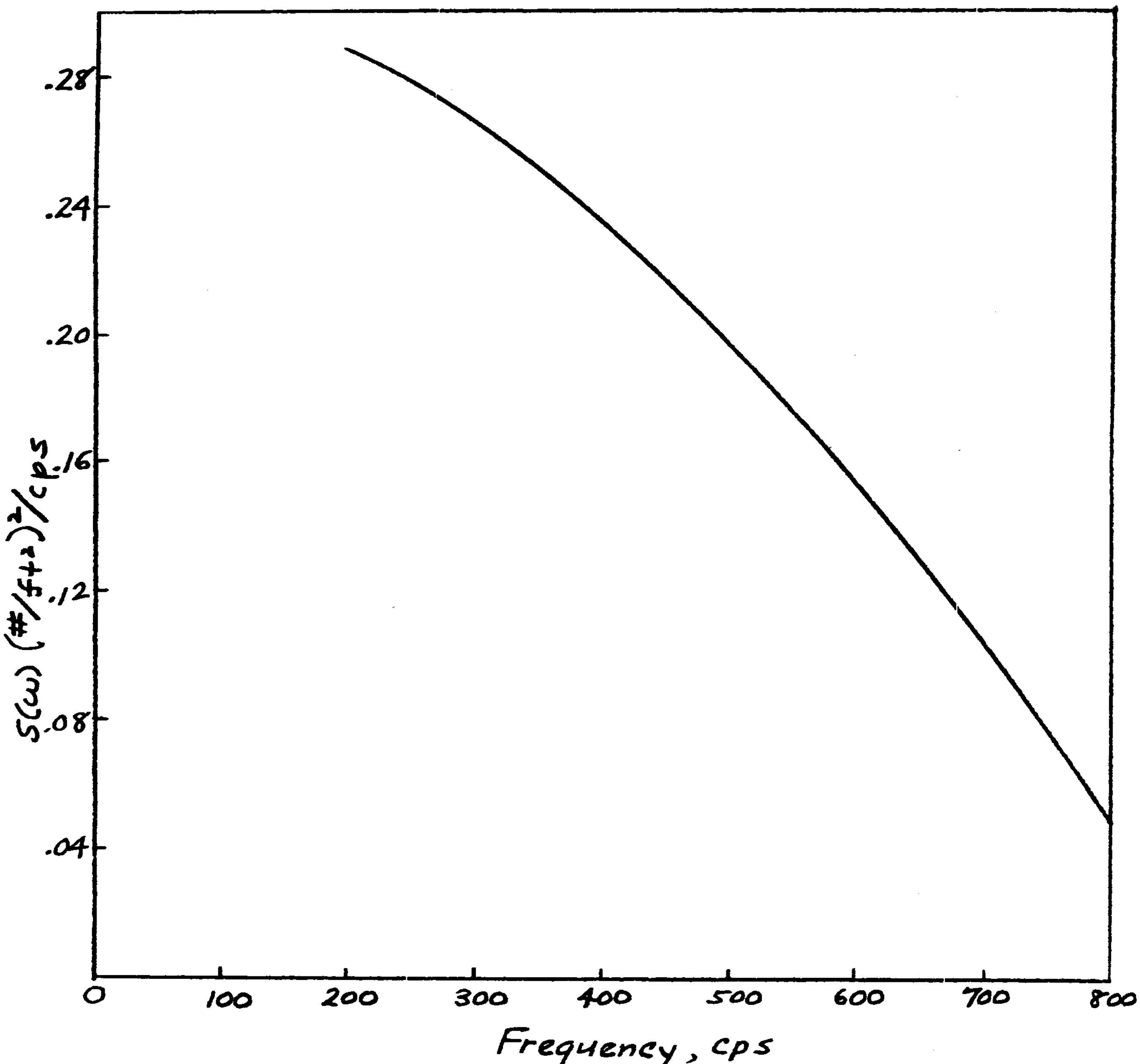

Fig. 2 Load Spectrum

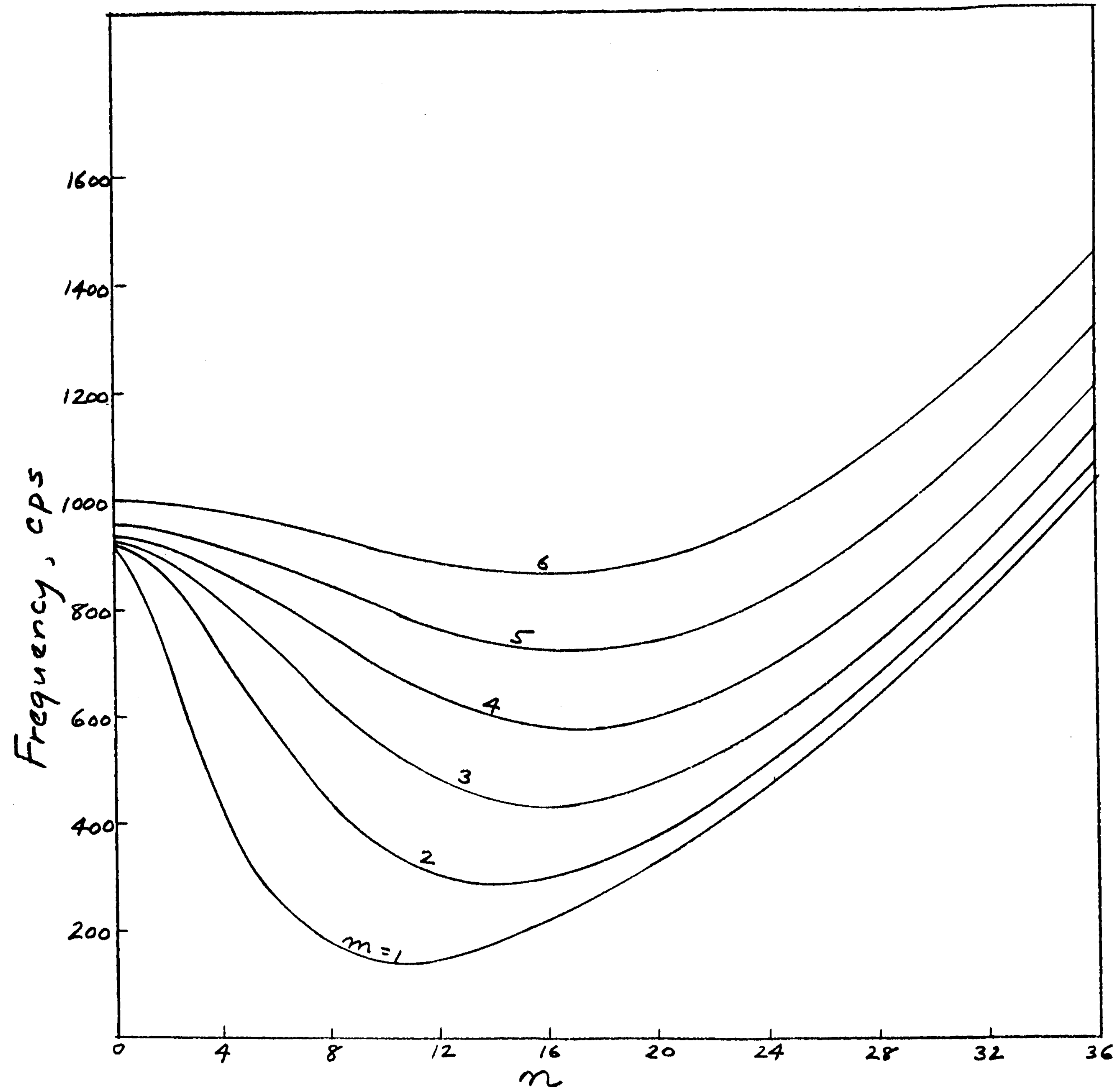

Fig. 3 Theoretical Frequency Spectrum
Unpressurized

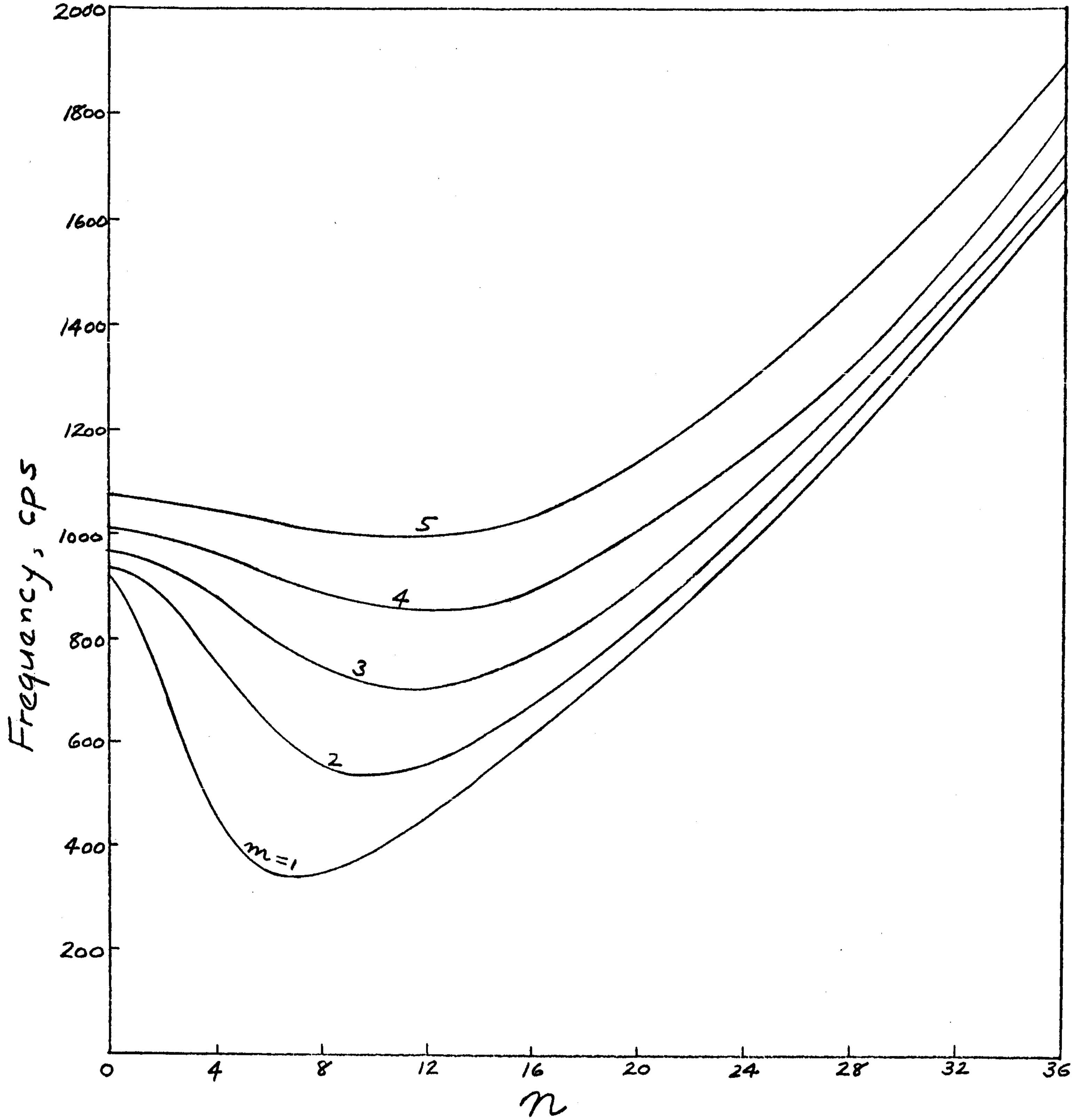

Fig. 4 Theoretical Frequency Spectrum Pressurized

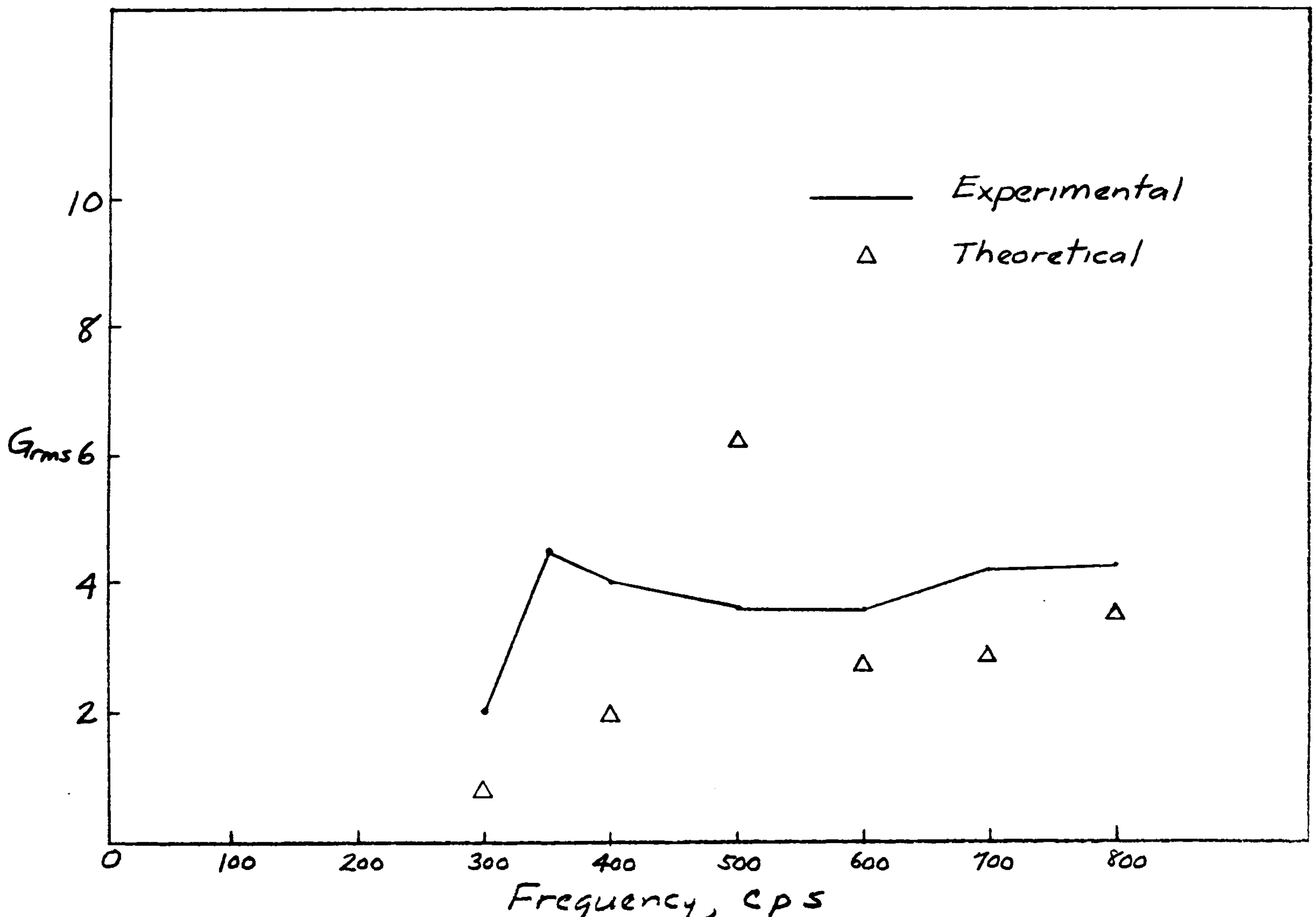

Fig. 5 Comparison of Theory and Experiment-
Empty Pressurized Shell
(Bandwidth = 10 cps)

Addendum to Report

(Page numbers refer to original report)

p. 7 In equation [9] each bracket along the diagonal should have a $-\Omega^2$ term in it; thus the A_{mn} bracket in the first equation should read (also changing signs of q_1 and q_2 to represent positive internal pressure)

$$A_{mn}\left[\lambda^2+\frac{1-\nu}{2}n^2\left(1+\frac{h^2}{12a^2}\right)+q_1 n^2+q_2\lambda^2-\Omega^2\right]$$

the B_{mn} bracket in the second equation should read

$$B_{mn}\left[n^2+\frac{1-\nu}{2}\lambda^2\left(1+3\frac{h^2}{12a^2}\right)+q_1 n^2+q_2\lambda^2-\Omega^2\right]$$

and the C_{mn} bracket in the third equation should read

$$C_{mn}\left[1+\frac{h^2}{12a^2}(\lambda^4+2\lambda^2n^2+n^4-2n^2+1)+q_1 n^2+q_2\lambda^2-\Omega^2\right]$$

p. 7 In eq. [9] change the signs of all terms involving q_1 and q_2

p. 8 the heading "C" should read "C. Random time loading with assumed space distribution"

p. 8 The paragraph after eq. [16] should read

From this point on, all the well known theorems for random loading[1,11] can be applied to the above series. In particular if cross product terms are neglected the power spectral density of the response can be written

p. 8 Under equation [17] write

If cross product terms are included

$$S_w(\omega)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}A_{mn}A_{pq}\,S_p(\omega)\,|H_w(m,n,\omega)|\,|H_w(p,q,\omega)|\cdot\cos(\theta_{mn}-\theta_{pq}) \quad [17a]$$

p. 9 The heading "III" should read

III. Illustrative example - Saturn Lox Tank with assumed space distribution

p. 10 The statement immediately before Table 1 should read

The response was computed for 300, 400, 500, 600, 700, 800 cps neglecting cross product terms. The modes considered at each frequency are given below in Table 1

p. 11. Add the following sections to the report

IV. General Random Loading

A. Basic equations

Starting with eq. [13] let $t-\theta=u$

$$w(x,\phi,t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sin\frac{m\pi x}{l}\cos n\phi\int_{-\infty}^{+\infty}f_{mn}(t-u)\,h_w(m,n,u)\,du$$

$$=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sin\frac{m\pi x}{l}\cos n\phi\int_{-\infty}^{+\infty}h_w(m,n,u)\,du \cdot\frac{2}{\pi l}\int_0^l\int_0^{2\pi}f(\xi,\eta,\theta)\sin\frac{m\pi\xi}{l}\cos n\eta\,d\xi\,d\eta \qquad [18]$$

The correlation is

$$R_w(x_1,x_2,\phi_1,\phi_2,\theta_1,\theta_2)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\Big[\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sin\frac{m\pi x_1}{l}\cos n\phi_1\int_{-\infty}^{+\infty}h_w(m,n,u_1)\,du_1 \cdot\frac{2}{\pi l}\int_0^l\int_0^{2\pi}f(\xi_1,\eta_1,\theta_1)\sin\frac{m\pi\xi_1}{l}\cos n\eta_1\,d\xi_1\,d\eta_1\Big]$$

$$\cdot\Big[\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\sin\frac{p\pi x_2}{l}\cos q\phi_2\int_{-\infty}^{+\infty}h_w(p,q,u_2)\,du_2 \cdot\frac{2}{\pi l}\int_0^l\int_0^{2\pi}f(\xi_2,\eta_2,\theta_2)\sin\frac{p\pi\xi_2}{l}\cos q\eta_2\,d\xi_2\,d\eta_2\Big]dt \qquad [19]$$

The power spectral density of the deflection w will be

$$S_w(x_1,\phi_1,\omega)=\sum_m\sum_n\sum_p\sum_q\Big[\sin\frac{m\pi x_1}{l}\sin\frac{p\pi x_1}{l}\cos n\phi_1\cos q\phi_1 \cdot\int_{-\infty}^{+\infty}e^{-i\omega u_1}h_w(m,n,u_1)\,du_1 \cdot\int_{-\infty}^{+\infty}e^{i\omega u_2}h_w(p,q,u_2)\,du_2$$

$$\cdot\frac{4}{\pi^2 l^2}\int_0^l\int_0^l\int_0^{2\pi}\int_0^{2\pi}S_f(\xi_1,\eta_1,\xi_2,\eta_2,\omega)\sin\frac{m\pi\xi_1}{l}\sin\frac{p\pi\xi_2}{l}\cos n\eta_1\cos q\eta_2\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2 \qquad [20]$$

where S_f is the power spectrum of the load

Assume S_f can be written

$$S_f(\xi_1, \eta_1, \xi_2, \eta_2, \omega) = \bar{S}_f(\omega) A(\xi_1 - \xi_2) B(\eta_1 - \eta_2) \qquad [21]$$

Then

$$S_w(x_1, \phi_1, \omega) = \sum_m \sum_n \sum_p \sum_q \sin\frac{m\pi x_1}{\ell} \sin\frac{p\pi x_1}{\ell} \cos n\phi_1 \cos q\phi_1 \; |H_w(m,n,\omega)| \, |H_w(p,q,\omega)| \cos(\theta_{mn} - \theta_{pq})$$
$$\cdot \frac{4}{\pi^2 \ell^2} \left[\bar{S}_f(\omega)\right] \left[\int_0^\ell \int_0^\ell \int_0^{2\pi} \int_0^{2\pi} A(\xi_1 - \xi_2) B(\eta_1 - \eta_2) \sin\frac{m\pi\xi_1}{\ell} \sin\frac{p\pi\xi_2}{\ell} \cos n\eta_1 \cos q\eta_2 \cdot d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2\right.$$

[22]

where θ_{mn} and θ_{pq} are the phase angles associated with the response functions of these modes.

The average power spectral density is found by integrating over the area of the cylinder. Thus

$$(S_w)_{ave} = \sum_m \sum_n \frac{1}{\pi^2 \ell^2} \bar{S}_f(\omega) \, |H_w(m,n,\omega)|^2$$
$$\cdot \int_0^\ell \int_0^\ell \int_0^{2\pi} \int_0^{2\pi} A(\xi_1 - \xi_2) B(\eta_1 - \eta_2) \sin\frac{m\pi\xi_1}{\ell} \sin\frac{m\pi\xi_2}{\ell} \cos n\eta_1 \cos n\eta_2 \; d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2 \qquad [23]$$

B. Application to Saturn Lox Tank

If the correlation function is assumed to be unity in both the circumferential and longitudinal directions and if the correlation is zero over the acoustically shaded area, the power spectral density of the deflection is @ $\phi_1 = 0$

$$S_w(x_1, \phi_1, \omega) = \sum_m \sum_n \sum_p \sum_q \sin\frac{m\pi x_1}{\ell} \sin\frac{p\pi x_1}{\ell} \, |H_w(m,n,\omega)| \, |H_w(p,q,\omega)| \cos(\theta_{mn} - \theta_{pq}) \frac{64}{mnpq\pi^4}$$

[24]

The power spectral density of the acceleration is determined by multiplying the above equation by ω^4. This result is identical to the result obtained in Section C of the report for assumed space distributions.

Fig. 5a shows theoretical-experimental comparisons using eq. [17] of section C and using eq. [24]. The results indicate that product terms do have significance in this case. There are some serious questions concerning the use of unit correlations since experimental evidence has shown that the space correlations in actual missiles in flight are much more complex. The results do indicate, however, that order of magnitude results can be obtained from these simplifying assumptions.

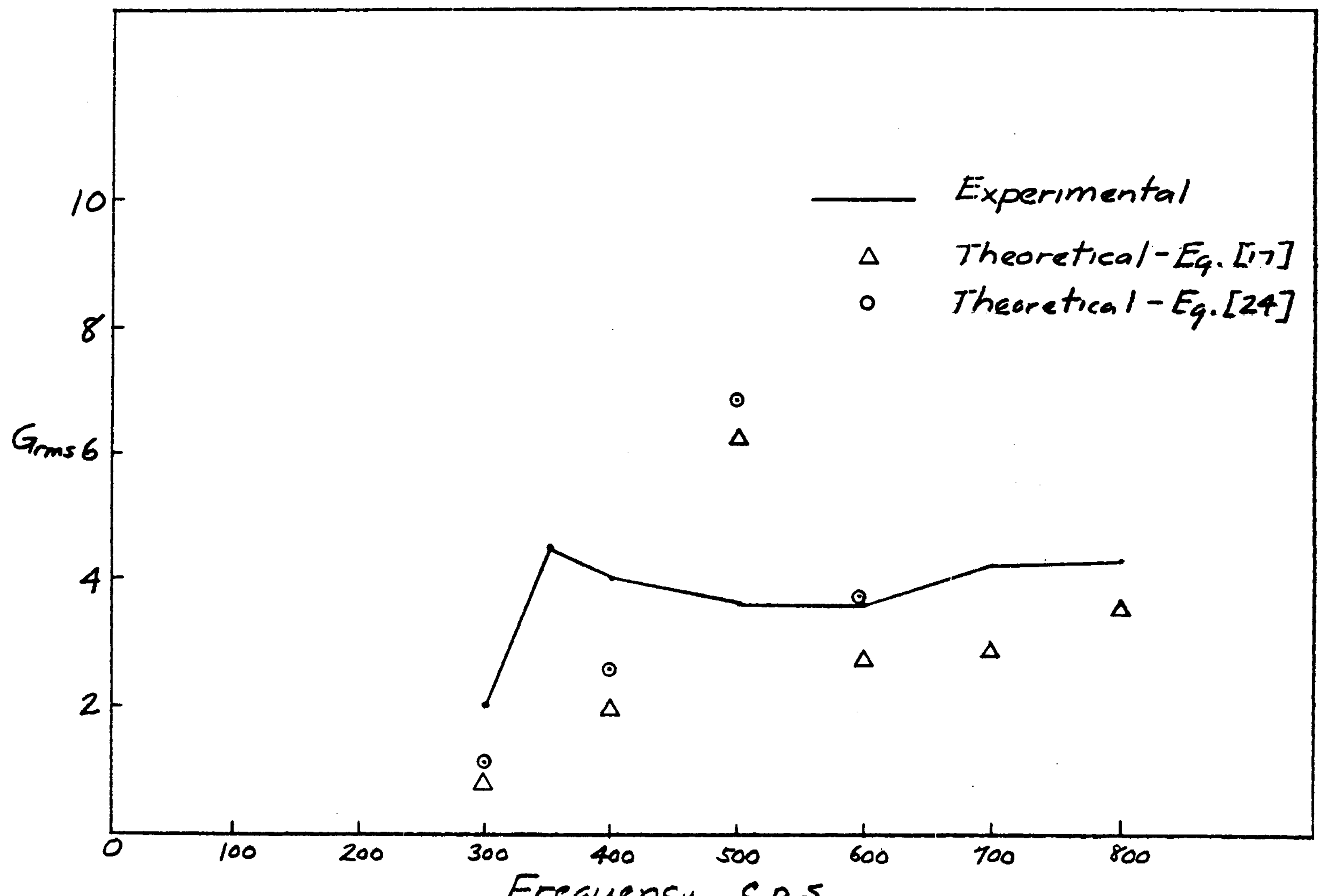

Fig. 5a Comparison of Theory and Experiment-
Empty Pressurized Shell
(Bandwidth = 10 cps)